

ROOT PARAMETRIZED DIFFERENTIAL EQUATIONS

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ABSTRACT. Let $C\langle \mathbf{t} \rangle$ be the differential field generated by l differential indeterminates $\mathbf{t} = (t_1, \dots, t_l)$ over an algebraically closed field C of characteristic zero. In this article we present almost generic normal forms $L(y, \mathbf{t}) \in C\langle \mathbf{t} \rangle\{y\}$ for several classical groups of Lie rank l .

1. INTRODUCTION

Let C be an algebraically closed field of characteristic zero and consider the differential field $F_1 = C\langle \mathbf{t} \rangle$ which is differentially generated by the l differential indeterminates $\mathbf{t} = (t_1, \dots, t_l)$ over C . Denote by \mathcal{G} one of the classical groups of Lie type A_l, B_l, C_l, D_l or G_2 (if $l = 2$) over C and let n be the dimension of the canonical representation of $\mathcal{G}(C)$. In this setting we present an explicit linear parameter differential equation $L(y, \mathbf{t}) = \sum_{i=0}^n a_i(\mathbf{t})y^{(i)} \in F_1\{y\}$ over F_1 which has $\mathcal{G}(C)$ as differential Galois group. Our method for the realization of \mathcal{G} uses the geometrical structure of \mathcal{G} as a semisimple linear algebraic group and transfers it to our differential equation $L(y, \mathbf{t})$. In this spirit the equation $L(y, \mathbf{t})$ defines a large family of differential equations with differential Galois group $\mathcal{G}(C)$ and we consider $L(y, \mathbf{t})$ as an almost generic equation for \mathcal{G} .

In literature there can be found two principle types of solutions for the generic inverse problem in differential Galois theory. For example in [2] Goldmann showed that generic differential equations for some of the above groups can be constructed. He used similar ideas as E. Noether for polynomial equations in classical Galois theory. More precisely, Goldmann takes the differential field of n generic solutions as his extension field, i.e., he starts with a differential field which contains n differential transcendental elements y_1, \dots, y_n and considers then the fixed field under $\mathcal{G}(C)$ whereas the action of $\mathcal{G}(C)$ is induced by matrix multiplication on the Wronskian $W(y_1, \dots, y_n)$. Goldmann proved that it is necessary for a generic differential equation in his sense to exist that the differential transcendence degree of the fixed field is n . In [4] L. Juan and A. Ledet pursued another method for the determination of generic equations. Their method is based on Kolchin's Structure Theorem which describes all possible Picard Vessiot extensions as function fields of irreducible \mathcal{G} -Torsors. In the case of SO_n their method is well applicable and yields a generic matrix differential equation with $\frac{1}{2}(n+2)(n-1)$ parameters (see [4]). The following theorem contains our parameter equations for the above groups of Lie type.

Theorem 1.1. *The linear parameter differential equation*

- (1) $L(y, \mathbf{t}) = y^{(l+1)} - \sum_{i=1}^l t_i y^{(i-1)} = 0$ has $SL_{l+1}(C)$ as differential Galois group over F_1 .
- (2) $L(y, \mathbf{t}) = y^{(2l)} - \sum_{i=1}^l (-1)^{i-1} (t_i y^{(l-i)})^{(l-i)} = 0$ has $SP_{2l}(C)$ as differential Galois group over F_1 .

- (3) $L(y, \mathbf{t}) = y^{(2l+1)} - \sum_{i=1}^l (-1)^{i-1} ((t_i y^{(l+1-i)})^{(l-i)} + (t_i y^{(l-i)})^{(l+1-i)}) = 0$
has $\mathrm{SO}_{2l+1}(C)$ as differential Galois group over F_1 .
- (4) $L(y, \mathbf{t}) = y^{(2l)} - 2 \sum_{i=3}^l (-1)^i ((t_i y^{(l-i)})^{(l+2-i)} + (t_i y^{(l+1-i)})^{(l+1-i)}) -$
 $(t_2 y^{(l-2)} + t_1 y)^{(l)} - ((-1)^l t_1 z_1 + z_2) - \sum_{i=0}^{l-2} (t_2^{l-2-i} z_1)^{(i)} = 0$ has $\mathrm{SO}_{2l}(C)$
as differential Galois group over F_1 . The substitutions z_1 and z_2 can be
found in Chapter 7.3 of [9].
- (5) $L(y, \mathbf{t}) = y^{(7)} + 2t_1 y' + 2(t_1 y)' + 2(t_2 y^{(4)})' + (t_2 y')^{(4)} - 2(t_2(t_2 y')')' = 0$
has $\mathrm{G}_2(C)$ as differential Galois group over F_1 .

Until now we did not yet study the generic properties of the equations in Theorem 1.1 in detail. But we will present in Section 5 a proof of the following addendum:

Addendum: *In case the differential ground field is algebraically closed the equation in Theorem 1.1 for $\mathrm{SL}_{l+1}(C)$ is a generic equation.*

Thus the above addendum gives a differential analogue for the group $\mathrm{SL}_{l+1}(C)$ to the Kummer equations for regular cyclic extensions in classical Galois theory. In particular, our equations for $\mathrm{SL}_{l+1}(C)$ are relatively simple differential equations which become generic after a suitable algebraic extension.

So far we do not know if similar results for the remaining groups in Theorem 1.1 hold. But we hope to come back to this questions in future publications.

For all subsequent sections, C denotes an algebraically closed field of characteristic zero and we keep the definition of the differential field F_1 .

2. BOUNDS FOR THE DIFFERENTIAL GALOIS GROUP

In this and the following section we present our method for the realization of the classical groups as differential Galois groups.

For this purpose let \mathcal{G} be one of the classical groups of Lie rank l over C and let F be a differential field with constants C . The main idea for the realization of $\mathcal{G}(C)$ is the definition an appropriate differential structure on a finite dimensional F vector space M such that the differential Galois group of M can not be larger or smaller than $\mathcal{G}(C)$. The differential structure on M is defined by a matrix differential equation $\partial(\mathbf{y}) = A\mathbf{y}$ where $A \in \mathrm{M}_n(F)$ and n is the dimension of a representation of $\mathcal{G}(C)$. As in [7] we apply an upper and lower bound criterion to A and choose $A \in \mathrm{M}_n(F)$ such that both bounds coincide. The main ingredient for a successful choice of A is the Lie algebra $\mathrm{Lie}(\mathcal{G})$ of \mathcal{G} .

An upper bound criterion for the differential Galois group is given by the following Proposition (see [8]).

Proposition 2.1. *Let \mathcal{H} be a connected linear algebraic group over C and let $A \in \mathrm{Lie}(\mathcal{H})(F)$. Then the differential Galois group $\mathcal{G}(C)$ of the differential equation $\partial(\mathbf{y}) = A\mathbf{y}$ is contained (up to conjugation) in $\mathcal{H}(C)$.*

Let R be a Picard Vessiot ring for $\partial(\mathbf{y}) = A\mathbf{y}$ with Galois group \mathcal{G} . Then the affine group scheme $\mathrm{Spec}(R) = \mathcal{Z}$ over F is a \mathcal{G} -torsor (see [8]). If \mathcal{Z} has a F -rational point, i.e., \mathcal{Z} is the trivial torsor, then Proposition 2.1 has a partial converse.

Proposition 2.2. *Let R be a Picard Vessiot ring for $\partial(\mathbf{y}) = A\mathbf{y}$ over F with connected differential Galois group $\mathcal{G}(C)$ and let \mathcal{Z} be the associated torsor. Let $\mathcal{H}(C) \supset \mathcal{G}(C)$ be a connected linear algebraic group with $A \in \mathrm{Lie}(\mathcal{H})(F)$. If \mathcal{Z} is the trivial torsor, then there exists $B \in \mathcal{H}(F)$ such that $\partial(B)B^{-1} + BAB^{-1}$ is an element of $\mathrm{Lie}(\mathcal{G})(F)$.*

For a proof see [8].

The condition of Proposition 2.2 is automatically satisfied if the cohomological dimension of F is almost one. From [10] we know that this is true for $F_2 := C(z)$, i.e., the function field with standard derivation $\frac{d}{dz}$. In this setting C. Mitchi and M. Singer found a way in [7] to apply successfully Proposition 2.1 and 2.2 for a realization of a connected semisimple group $\mathcal{G}(C)$.

Since the cohomological dimension of the differential field F_1 is not almost one, we have no information if in our setting the condition of Proposition 2.2 is satisfied and therefore we can not use it as a lower bound criterion. But we can apply it in an indirect way. Let $\partial(\mathbf{y}) = A(\mathbf{t})\mathbf{y}$ be a matrix differential equation with defining matrix $A(\mathbf{t}) \in R_1^{n \times n}$ and consider now a surjective specialization $\sigma : R_1 \rightarrow R_2$ whereas R_1, R_2 are the differential subrings $R_1 := C\{\mathbf{t}\} \subset F_1$ and $R_2 := C[z] \subset F_2$. Then σ yields a new differential equation $\partial(\mathbf{y}) = A(\sigma(\mathbf{t}))\mathbf{y}$ over F_2 . Intuitively we would now assume that the differential Galois group of the specialized equation $\partial(\mathbf{y}) = A(\sigma(\mathbf{t}))\mathbf{y}$ is contained in the differential Galois group of the original equation. We introduce the so-called specialization bound.

Theorem 2.3. *Suppose the defining matrix $A(\mathbf{t})$ satisfies $A(\mathbf{t}) \in R_1^{n \times n}$. Then the differential Galois group of the specialized equation $\partial(\mathbf{y}) = A(\sigma(\mathbf{t}))\mathbf{y}$ over F_2 is a subgroup of the differential Galois group for $A(\mathbf{t})$.*

Sketch of proof (see [9] for details):

Since the defining matrix $A(\mathbf{t})$ satisfies $A(\mathbf{t}) \in R_1^{n \times n}$, the differential structure on $R_1[\mathrm{GL}_n]$ is well defined. For a maximal differential ideal I_1 in $R_1[\mathrm{GL}_n]$, we obtain a Picard Vessiot ring $R_1[\mathrm{GL}_n]/I_1$ and the differential Galois group $\mathcal{G}_1(C)$ of $R_1[\mathrm{GL}_n]/I_1$ stabilizes the ideal I_1 . Now we can extend the surjective specialization σ to a differential ring homomorphism $\sigma : R_1[\mathrm{GL}_n] \rightarrow R_2[\mathrm{GL}_n]$ by $A(\sigma(\mathbf{t}))$. Then the differential ideal I_1 specializes to a differential ideal $\sigma(I_1)$. At this point we assume that $\sigma(I_1)$ is a proper differential ideal of $R_2[\mathrm{GL}_n]$. We can choose now a maximal differential ideal I_2 such that $\sigma(I_1) \subseteq I_2$. This yields a Picard Vessiot ring $R_2[\mathrm{GL}_n]/I_2$ with differential Galois group $\mathcal{G}_2(C)$. Since $\mathcal{G}_2(C)$ has to stabilize $I_2 \supseteq \sigma(I_1)$, we would say intuitively that $\mathcal{G}_2(C)$ has to satisfy the same and even more conditions than $\mathcal{G}_1(C)$. For this reason $\mathcal{G}_2(C)$ should be contained in $\mathcal{G}_1(C)$. A complete proof is given in [9].

We specify Theorem 2.3 in the following Corollary.

Corollary 2.4. *Let $\partial(\mathbf{y}) = A(\mathbf{t})\mathbf{y}$ be a differential equation over F_1 with differential Galois group $\mathcal{G}(C)$ and suppose $A(\mathbf{t}) \in R_1^{n \times n}$. Let*

$$\sigma : R_1 \rightarrow R_2, \quad \mathbf{t} \mapsto \mathbf{f} = (f_1, \dots, f_l) \in R_2^l$$

be a surjective specialization. Then the differential Galois group of $\partial(\mathbf{y}) = A(\sigma(\mathbf{t}))\mathbf{y}$ over F_2 is a subgroup of $\mathcal{G}(C)$.

3. RESULTS FROM THE THEORY OF ALGEBRAIC GROUPS

Before we describe the choice of the defining matrix $A(\mathbf{t})$ in $\mathrm{Lie}(\mathcal{G})(R_1)$ in order to show that the above bounds coincide for it, we recall some structure theory about semisimple linear algebraic groups.

Let Φ be the root system of $\mathrm{Lie}(\mathcal{G})(C)$ and denote by $\Delta = \{\alpha_1, \dots, \alpha_l\}$ a basis of Φ . Further let

$$\mathrm{Lie}(\mathcal{G})(C) = \mathbf{H}(C) \oplus \bigoplus_{\alpha \in \Phi} \mathrm{Lie}(\mathcal{G})_{\alpha}(C)$$

be a Cartan decomposition for $\mathrm{Lie}(\mathcal{G})(C)$ with Cartan subalgebra $\mathbf{H}(C)$ in diagonal form and one dimensional root spaces $\mathrm{Lie}(\mathcal{G})_{\alpha}(C)$ for the roots $\alpha \in \Phi$. We choose

now a Chevalley basis

$$\{X_\alpha, H_{\alpha_i} \mid \alpha \in \Phi, \alpha_i \in \Delta\}$$

associated to the above Cartan decomposition for $\text{Lie}(\mathcal{G})(C)$.

We specify now the choice of the defining matrix $A(t)$. For the simple roots $\alpha_i \in \Delta$ we define the matrix $A_\Delta := \sum_{\alpha_i \in \Delta} X_{\alpha_i}$. Then there are l negative roots $-\gamma_i \in \Gamma^- \subset \Phi^-$ whose choice depends on the properties of the root system of \mathcal{G} , such that the matrix

$$A(t) := A_\Delta + \sum_{-\gamma_i \in \Gamma^-} t_i X_{-\gamma_i} \in \text{Lie}(\mathcal{G})(R_1)$$

does not lie in any subalgebra of $\text{Lie}(\mathcal{G})(F_1)$ and covers by specialization a wide range of differentially equivalent matrices (see below). In addition the differential equation $\partial(\mathbf{y}) = A(t)\mathbf{y}$ has a canonical cyclic vector which induces easily a linear parameter equation of a nice shape.

For a successful application of the specialization bound we need a differential equation $\partial(\mathbf{y}) = A\mathbf{y}$ over F_2 which is a specialization of the parameter equation $\partial(\mathbf{y}) = A(t)\mathbf{y}$ and has a known differential Galois group. For this reason it is useful to enlarge the amount of equations which are available as specializations of $A(t)$. For this purpose let $\partial(\mathbf{y}) = A\mathbf{y}$ be a differential equation over a differential field F with constants C and let M be the associated differential module. For a matrix $B \in \text{GL}_n(F)$ we can change the basis of M . This yields a so-called differentially or gauge equivalent equation

$$BAB^{-1} + \partial(B)B^{-1} = \tilde{A}.$$

Then by the geometrical structure of $\mathcal{G}(C)$ and the choice of the roots in the definition of $A(t)$, it is possible to show that every element in the subspace

$$(1) \quad A \in A_\Delta + \mathbf{H}(F) \bigoplus_{\alpha \in \Phi^-} \text{Lie}(\mathcal{G})_\alpha(F)$$

is differentially equivalent to a specialization of $A(t)$. More precisely, the adjoint action

$$\text{Ad}(B) : \text{Lie}(\mathcal{G}) \rightarrow \text{Lie}(\mathcal{G}), X \mapsto BXB^{-1} \text{ with } B \in \mathcal{G}(C)$$

and the logarithmic derivative

$$l\delta : \text{GL}_n(F) \rightarrow \text{Lie}(\text{GL}_n)(F), X \mapsto \partial_F(X)X^{-1}$$

yield that we can consider the differential transformation of A as the sum of the two maps $\text{Ad}(B)(A)$ and $l\delta(B)$, i.e., we have

$$BAB^{-1} + \partial(B)B^{-1} = \text{Ad}B(A) + l\delta(B).$$

In order to get a better grasp of the differential conjugation we need more information about the images of the two maps. To begin with we study the adjoint action. The Lie structure of $\mathcal{G}(C)$ helps us to control it. For $\beta \in \Phi$ and $x \in F$ let $U_\beta(x)$ be the image of the nilpotent derivation $x \text{ad}(X_\beta)$ under the exponential map. The unipotent elements $U_\beta(x)$ generate the so-called unipotent root groups $\mathcal{U}_\beta(x) \subset \mathcal{G}$. The benefit of $U_\beta(x)$ is that the effect of $\text{Ad}(\mathcal{U}_\beta(x))$ on the elements of a Chevalley basis can be described by the root system Φ . For details related to the following remark we refer to [3].

Remark 1. For $\alpha, \beta \in \Phi$ linearly independent let $\alpha - r\beta, \dots, \alpha + q\beta$ be the β -string through α ($r, q \in \mathbb{N}$) and let $\langle \alpha, \beta \rangle$ be the Cartan integer. We define $c_{\beta, \alpha, 0} := 0$ and $c_{\beta, \alpha, i} := \pm \binom{r+i}{i}$. Then we have

$$\begin{aligned} \text{Ad}(U_\beta(x))(X_\alpha) &= \sum_{i=0}^q c_{\beta, \alpha, i} x^i X_{\alpha+i\beta} \\ \text{Ad}(U_\beta(x))(H_\alpha) &= H_\alpha - \langle \alpha, \beta \rangle X_\beta \end{aligned}$$

and

$$\text{Ad}(U_\beta(x))(X_{-\beta}) = X_{-\beta} + xH_\beta - x^2X_\beta.$$

The following remark allows us to describe the image of the elements of the root groups under the logarithmic derivative during the differential transformation of A . A proof can be found in [5].

Remark 2. Let $\mathcal{G} \subset \text{GL}_n$ be a linear algebraic group. Then the restriction

$$l\delta|_{\mathcal{G}}: \mathcal{G}(F) \rightarrow \text{Lie}(\mathcal{G})(F)$$

maps $\mathcal{G}(F)$ to its Lie algebra.

At this point we want to note that N. Elkies refers in [1] exactly to the subspace in (1). More precisely, he uses the subspace $A_\Delta + \text{Lie}(\mathcal{B})$, whereas \mathcal{B} denotes here the Borel subgroup in lower triangular form of \mathcal{G} , to define a *subvariety* \mathcal{X} of the flag manifold \mathcal{G}/\mathcal{B} and proposes it as a differential analogue of the Deligne-Lusztig variety.

Our discussion showed that the key ingredient for the proof of Theorem 1.1 is the geometrical structure of the underlying Lie group. For this reason the proofs for the single groups are very similar and we present in the next section exemplarily the proof for the group \mathcal{G} of type A_l , i.e., the special linear group $\text{SL}_{l+1}(C)$. Proofs for the remaining groups of type B_l , C_l , D_l and G_2 can be found in [9].

4. AN EXEMPLARY PROOF FOR $\text{SL}_{l+1}(C)$

In this section we carry out explicitly the proof of Theorem 1.1 for the special linear group $\text{SL}_{l+1}(C)$.

As we have seen in the previous section an important object for the realization of $\text{SL}_{l+1}(C)$ is the root system. For this reason let Φ be the root system of type A_l and denote by $\Delta = \{\alpha_1, \dots, \alpha_l\}$ a basis of Φ . With respect to Δ the root system $\Phi = \Phi^+ \cup \Phi^-$ consists of the elements

$$\Phi^+ = \{\alpha_s + \dots + \alpha_t \mid 1 \leq s \leq t \leq l\} \text{ and } \Phi^- = \{-\alpha \mid \alpha \in \Phi^+\}.$$

Note that for the following it is helpful to have the shape of the roots of Φ in mind. The Dynkin diagram for A_l implies that we can decompose Φ for $1 \leq k \leq l$ in subsystems Φ_k of type A_k with basis

$$\Delta_k = \{\alpha_{l-k+1}, \dots, \alpha_l\}.$$

Since Φ_k is a root system, the set of positive roots Φ_k^+ of Φ_k has a unique root of maximal height (e.g. see [3]) which we denote by γ_k . For $0 \leq k \leq l-1$ we denote by

$$\Gamma_k := \{\gamma_i \mid \gamma_i \text{ is the maximal root of } \Phi_i^+, k+1 \leq i \leq l\}$$

the set of maximal roots of the descending subsystems $\Phi = \Phi_l \supseteq \dots \supseteq \Phi_{k+1}$. Further, we define $\Gamma_l := \emptyset$ and we write shortly Γ for Γ_0 . By Γ_k^- we denote the same set as Γ_k only for the negative roots. Later we will use the roots of Γ^- for our parametrization.

Remark 3. From the shapes of the roots in Φ^+ we deduce that $\Phi_k^+ \setminus \Phi_{k-1}^+$ is the set of roots

$$\Phi_k^+ \setminus \Phi_{k-1}^+ = \{\alpha_{l-k+1} + \dots + \alpha_{l-k+m} \mid 1 \leq m \leq k\}$$

for $k \in \{1, \dots, l\}$ and that Φ^+ is the disjoint union of all the $\Phi_k^+ \setminus \Phi_{k-1}^+$. Here Φ_0 denotes the empty set. We obtain further that for any element m of $\{1, \dots, k\}$ there is a unique root α with $\text{ht}(\alpha) = m$ such that α is contained in $\Phi_k^+ \setminus \Phi_{k-1}^+$. A formal proof uses two inductions, i.e., an induction on the subsystems Φ_k and an inner induction on the height m of the roots in $\Phi_k^+ \setminus \Phi_{k-1}^+$.

Remark 4. Suppose $k \in \{1, \dots, l\}$ and $m \in \{1, \dots, k\}$. Then Remark 3 implies that for a root $\alpha \in \Phi_k^+ \setminus (\Phi_{k-1}^+ \cup \{\gamma_k\})$ with $\text{ht}(\alpha) = m$ there exists a unique simple root $\alpha_s \in \Delta$ such that $\beta := \alpha + \alpha_s \in \Phi_k^+ \setminus \Phi_{k-1}^+$. In particular, if for $\alpha_s \in \Delta$ the sum $\beta - \alpha_s$ is a root, then either $\beta - \alpha_s = \alpha$ or $\beta - \alpha_s \in \Phi_{k-1}^+$.

Now we present an explicit Chevalley basis according to a Cartan decomposition of $\text{Lie}(\text{SL}_{l+1})(C)$. Let us denote by E_{ij} the matrix with 1 at position (i, j) and 0 elsewhere. Then the following matrices define a Chevalley basis for $\text{Lie}(\text{SL}_{l+1})(C)$ (for details we refer to [9]):

$$\begin{aligned} X_\alpha &:= E_{s,t+1} && \text{for } \alpha = \alpha_s + \dots + \alpha_t \in \Phi^+, \\ X_{-\alpha} &:= E_{t+1,s} && \text{for } -\alpha = -\alpha_s - \dots - \alpha_t \in \Phi^- \text{ and} \\ H_{\alpha_i} &:= X_{i,i} - X_{i+1,i+1} && \text{for } \alpha_i \in \Delta. \end{aligned}$$

The next step is the appropriate choice of the defining matrix $A(\mathbf{t}) \in \text{Lie}(\text{SL}_{l+1})(R_1)$ for our parameter equation. For this purpose let $A_\Delta := \sum_{\alpha_i \in \Delta} X_{\alpha_i}$ be defined as in the previous section with the explicit matrices X_{α_i} from above. We define

$$A(\mathbf{t}) := A_\Delta + \sum_{-\gamma_i \in \Gamma^-} t_i X_{-\gamma_i} \in \text{Lie}(\text{SL}_{l+1})(R_1).$$

The roots of Γ^- and the simple roots α_i imply that $A(\mathbf{t})$ is not contained in any proper subalgebra of $\text{Lie}(\text{SL}_{l+1})(F_1)$. Additionally, the choice of the above Chevalley basis yields that the matrix $A(\mathbf{t})$ has the shape of a companion matrix, i.e.,

$$A(\mathbf{t}) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & \dots & & 0 & 1 \\ t_1 & t_2 & \dots & t_l & 0 \end{pmatrix}$$

and induces therefore the easy and nice linear differential equation of Theorem 1.1 for $\text{SL}_{l+1}(C)$. Note that for similar choices of $A(\mathbf{t})$ for the groups of type B_l , C_l , D_l and G_2 the equation $\partial(\mathbf{y}) = A(\mathbf{t})\mathbf{y}$ has also a canonical cyclic vector although $A(\mathbf{t})$ is not a companion matrix anymore. To complete the proof for $\text{SL}_{l+1}(C)$ we need to show that the two bounds for $A(\mathbf{t})$ coincide.

Since we intend to apply Corollary 2.4, we need a differential equation $\partial(\mathbf{y}) = \bar{A}\mathbf{y}$ which has $\text{SL}_{l+1}(C)$ as differential Galois group and whose defining matrix \bar{A} satisfies $\bar{A} \in \text{Lie}(\text{SL}_{l+1})(R_2)$ and $\bar{A} = \sigma(A(\mathbf{t}))$ for a specialization $\sigma : R_1 \rightarrow R_2$. The following proposition shows that we have access to a large class of equations.

Proposition 4.1. *Suppose the matrix A satisfies*

$$A \in A_\Delta + \mathbf{H}(F) \oplus \bigoplus_{\delta \in \Phi^-} \text{Lie}(\text{SL}_{l+1})_\delta(F).$$

Then A is differentially equivalent to a matrix in the subspace

$$A_\Delta + \bigoplus_{-\gamma_i \in \Gamma^-} \text{Lie}(\text{SL}_{l+1})_{-\gamma_i}(F).$$

The proof uses Lemma 4.2 and 4.3 from below.

Lemma 4.2. *Suppose that the matrix A satisfies*

$$A \in A_\Delta + \mathbf{H}(F) \oplus \bigoplus_{\delta \in \Phi^-} \text{Lie}(\text{SL}_{l+1})_\delta(F).$$

Then there exists $U \in \mathcal{U}^-(F)$ such that the gauge equivalent matrix $\text{Ad}(U)(A) + l\delta(U)$ is an element of the subspace

$$A_\Delta + \bigoplus_{\delta \in \Phi^-} \text{Lie}(\text{SL}_{l+1})_\delta(F).$$

Proof. For $j \in \{1, \dots, l\}$ let A be the matrix

$$A = A_\Delta + \sum_{j \leq i \leq l} h_i H_i + \sum_{\delta \in \Phi^-} Z_\delta$$

with $h_i \in F$ and $Z_\delta \in \text{Lie}(\text{SL}_{l+1})_\delta(F)$. In the first step we delete the part $h_j H_j$ of A . For this purpose, we consider the image of A_Δ and $\sum_{j \leq i \leq l} h_i H_i$ under $\text{Ad}(U_{-\alpha_j}(x))$. Remark 1 yields

$$\text{Ad}(U_{-\alpha_j}(x))(A_\Delta) \in A_\Delta + xH_j \oplus \text{Lie}(\text{SL}_{l+1})_{-\alpha_j}(F)$$

and

$$\text{Ad}(U_{-\alpha_j}(x))\left(\sum_{j \leq i \leq l} h_i H_i\right) \in \sum_{j \leq i \leq l} h_i H_i \oplus \text{Lie}(\text{SL}_{l+1})_{-\alpha_j}(F).$$

Further, the subspace $\bigoplus_{\delta \in \Phi^-} \text{Lie}(\text{SL}_{l+1})_\delta(F)$ is stabilized by $\text{Ad}(U_{-\alpha_j}(x))$, i.e.,

$$\text{Ad}(U_{-\alpha_j}(x))\left(\sum_{\delta \in \Phi^-} Z_\delta\right) \in \bigoplus_{\delta \in \Phi^-} \text{Lie}(\text{SL}_{l+1})_\delta(F).$$

Remark 2 implies $l\delta(U_{-\alpha_j}(x)) \in \text{Lie}(\text{SL}_{l+1})_{-\alpha_j}(F)$. If we sum up our results and define $x := -h_j$ we obtain finally

$$\text{Ad}(U_{-\alpha_j}(x))(A) \in A_\Delta + \sum_{j+1 \leq i \leq l} h_i H_i + \bigoplus_{\delta \in \Phi^-} \text{Lie}(\text{SL}_{l+1})_\delta(F).$$

Now an induction shows that A is differentially equivalent to a matrix in the subspace

$$A_\Delta + \bigoplus_{\delta \in \Phi^-} \text{Lie}(\text{SL}_{l+1})_\delta(F).$$

□

Lemma 4.3. Let k be an element of $\{1, \dots, l\}$ and suppose the matrix A satisfies

$$A \in A_\Delta + \bigoplus_{-\gamma_i \in \Gamma_k^-} \text{Lie}(\text{SL}_{l+1})_{-\gamma_i}(F) \oplus \bigoplus_{\delta \in \Phi_k^-} \text{Lie}(\text{SL}_{l+1})_\delta(F).$$

Then there exists $U \in \mathcal{U}^-(F)$ such that $\text{Ad}(U)(A) + l\delta(U)$ is an element of the subspace

$$A_\Delta + \bigoplus_{-\gamma_i \in \Gamma_{k-1}^-} \text{Lie}(\text{SL}_{l+1})_{-\gamma_i}(F) \oplus \bigoplus_{\delta \in \Phi_{k-1}^-} \text{Lie}(\text{SL}_{l+1})_\delta(F).$$

Proof. In the following we denote by $\Omega_{k,j}$ the set of roots $\alpha \in \Phi^-$ which satisfy $\alpha \in \Phi_k^- \setminus \Phi_{k-1}^-$ and $\text{ht}(\alpha) \geq j$. To begin with, we consider for $j \in \{1, \dots, k-1\}$ the matrix

$$A = A_\Delta + \sum_{-\gamma_i \in \Gamma_k^-} Z_{-\gamma_i} + \sum_{\delta \in \Omega_{k,j}} Z_\delta + \sum_{\delta \in \Phi_{k-1}^-} Z_\delta.$$

Here for $\delta \in \Phi^-$, Z_δ denotes an element of the root space $\text{Lie}(\text{SL}_{l+1})_\delta(F)$. We remove the part Z_δ of A where δ satisfies $\delta \in \Phi_k^- \setminus \Phi_{k-1}^-$ and $\text{ht}(\delta) = j$. By Remark 4 a root with this property is unique and we denote it by α . From Remark 4 we also know that there is a unique simple $\alpha_s \in \Delta$ such that $-\alpha + \alpha_s \in \Phi_k^+ \setminus \Phi_{k-1}^+$. Thus,

for $\beta := \alpha - \alpha_s$ we obtain $\beta + \alpha_s = \alpha$ and if for $\bar{\alpha}_s \in \Delta$ with $\bar{\alpha}_s \neq \alpha_s$ the sum $\beta + \bar{\alpha}_s$ is a root, then $\beta + \bar{\alpha}_s$ is an element of Φ_{k-1}^- . This and Remark 1 implies

$$\text{Ad}(U_\beta(x))(A_\Delta) \in A_\Delta + x c_{\beta, \alpha_s, 1} X_\alpha + \bigoplus_{\delta \in \Phi_{k-1}^-} \text{Lie}(\text{SL}_{l+1})_\delta(F).$$

Since $-\gamma_i$ is the maximal root of Φ_i^- for $k+1 \leq i \leq l$, at least one of the coefficients in $-\gamma_i + q\beta$ would be greater 1 for $q \in \mathbb{N} \setminus \{0\}$ and therefore $-\gamma_i + q\beta$ can not be a root of Φ . This yields that the subspace $\bigoplus_{-\gamma_i \in \Gamma_k^-} \text{Lie}(\text{SL}_{l+1})_{-\gamma_i}(F)$ is invariant under $\text{Ad}(U_\beta(x))$, i.e., we have

$$\text{Ad}(U_\beta(x))\left(\sum_{-\gamma_i \in \Gamma_k^-} Z_{-\gamma_i}\right) = \sum_{-\gamma_i \in \Gamma_k^-} Z_{-\gamma_i}.$$

A similar argumentation yields

$$\text{Ad}(U_\beta(x))\left(\sum_{\delta \in \Omega_{k,j}} Z_\delta\right) = Z_\alpha + \sum_{\delta \in \Omega_{k,j+1}} Z_\delta.$$

If for $q \in \mathbb{N} \setminus \{0\}$ and $\delta \in \Phi_{k-1}^-$ the sum $\delta + q\beta$ is a root of Φ , then it is an element of $\Omega_{k,j+2}$, i.e.,

$$\text{Ad}(U_\beta(x))\left(\sum_{\delta \in \Phi_{k-1}^-} Z_\delta\right) \in \sum_{\delta \in \Phi_{k-1}^-} Z_\delta + \bigoplus_{\delta \in \Omega_{k,j+2}} \text{Lie}(\text{SL}_{l+1})_\delta(F).$$

Finally, Remark 2 implies that the logarithmic derivative of $U_\beta(x)$ is an element of $\text{Lie}(\text{SL}_{l+1})_\beta(F)$. If we choose now $x \in F$ such that $xc_{\beta, \alpha_s, 1} X_\alpha = -Z_\alpha$ holds and if we sum everything from above up, we get

$$\begin{aligned} & \text{Ad}(U_\beta(x))(A) \in \\ & A_\Delta + \sum_{-\gamma_i \in \Gamma_k^-} Z_{-\gamma_i} + \bigoplus_{\delta \in \Omega_{k,j+1}} \text{Lie}(\text{SL}_{l+1})_\delta(F) \oplus \bigoplus_{\delta \in \Phi_{k-1}^-} \text{Lie}(\text{SL}_{l+1})_\delta(F). \end{aligned}$$

Now an induction completes the proof. \square

Proof of Proposition 4.1: It can be shown by induction in combination with Lemma 4.2 that the matrix A is differentially equivalent to a matrix in the subspace

$$A_\Delta + \bigoplus_{\delta \in \Phi^-} \text{Lie}(\text{SL}_{l+1})_\delta(F) = A_\Delta + \bigoplus_{1 \leq k \leq l} \bigoplus_{\delta \in \Phi_k^- \setminus \Phi_{k-1}^-} \text{Lie}(\text{SL}_{l+1})_\delta(F),$$

i.e., we delete the part of A which lies in the Cartan subalgebra $\mathbf{H}(F)$. Now a second induction in combination with Lemma 4.3 yields the assertion of the proposition.

For a proof of the case for $\text{SL}_{l+1}(C)$ in Theorem 1.1 we need a matrix differential equation $\partial(\mathbf{y}) = \bar{A}\mathbf{y}$ over R_2 which has $\text{SL}_{l+1}(C)$ as differential Galois group and which satisfies $\bar{A} \in A_\Delta + \text{Lie}(\mathcal{B}^-)(R_2)$. Such an equation yields a variant of a result from C. Mitschi and M. Singer.

Proposition 4.4. *Let \mathcal{G} be a connected semisimple linear algebraic group and set $A_0 = \sum_{\alpha_i \in \Delta} (X_{\alpha_i} + X_{-\alpha_i})$. Then there exists $A_1 \in \mathbf{H}(C)$ such that the differential equation $\partial(\mathbf{y}) = (A_0 + A_1 z^2)\mathbf{y}$ over $C(z)$ has \mathcal{G} as differential Galois group.*

The proof of the original version can be found in [7]. The difference of the original version to Proposition 4.4 is that we modified the choice of the matrix A_0 . For a proof of Proposition 4.4 we refer to [9].

Proof of Theorem 1.1 for $\mathrm{SL}_{l+1}(\mathbf{C})$: We define a differential equation $\partial(\mathbf{y}) = A(\mathbf{t})\mathbf{y}$ by

$$A(\mathbf{t}) = A_\Delta + \sum_{-\gamma_i \in \Gamma^-} t_i X_{-\gamma_i} \in \mathrm{Lie}(\mathrm{SL}_{l+1})(R_1).$$

Proposition 2.1 shows that the Galois group $\mathcal{G}(C)$ of $A(\mathbf{t})$ is a subgroup of $\mathrm{SL}_{l+1}(C)$. On the other hand there exists $A_1 \in \mathbf{H}(C)$ by Corollary 4.4 such that the differential Galois group of $\partial(\mathbf{y}) = (A_0 + z^2 A_1)\mathbf{y}$ is $\mathrm{SL}_{l+1}(C)$. Further, Proposition 4.1 yields that $(A_0 + z^2 A_1)$ is differentially equivalent to a matrix \bar{A} in the subspace

$$A_\Delta + \bigoplus_{-\gamma_i \in \Gamma^-} \mathrm{Lie}(\mathrm{SL}_{l+1})_{-\gamma_i}(R_2).$$

For the specialization $\sigma : \mathbf{t} \mapsto (f_1, \dots, f_l)$ with $\bar{A} = A_\Delta + \sum_{-\gamma_i \in \Gamma^-} f_i X_{-\gamma_i}$ we obtain that the Galois group of $\sigma(A(\mathbf{t}))$ is $\mathrm{SL}_{l+1}(C)$. Then by Corollary 2.4 we have $\mathrm{SL}_{l+1}(C) \subset \mathcal{G}(C) \subset \mathrm{SL}_{l+1}(C)$. This yields $\mathcal{G}(C) = \mathrm{SL}_{l+1}(C)$. If we now take y_1 as a cyclic vector, we obtain a linear differential equation for $\mathrm{SL}_{l+1}(C)$ as claimed in Theorem 1.1. This completes the proof.

5. GENERIC PROPERTIES OF THE PARAMETER EQUATION FOR $\mathrm{SL}_{l+1}(C)$

In this section we proof the addendum from the introduction and a refined version of it for the parameter equation

$$L(y, \mathbf{t}) = y^{(l+1)} - \sum_{i=1}^l t_i y^{(i-1)} = 0$$

of $\mathrm{SL}_{l+1}(C)$. For this purpose, let F be a differential field with field of constants C and let E/F be Picard Vessiot extension with differential Galois group $\mathrm{SL}_{l+1}(C)$ and defining matrix $A \in \mathrm{M}_{l+1}(F)$. Note that we can always find such a defining matrix by Theorem 2.33 of [8]. We want to answer the question whether there exists a specialization $\sigma : F_1 \rightarrow F$ such that the specialized equation $L(y, \sigma(\mathbf{t}))$ defines E/F ?

By the Cyclic Vector Theorem (e.g., see [6]), we can assume that A is a companion matrix, i.e., A has following shape:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & \dots & & 0 & 1 \\ a_1 & a_2 & \dots & & a_{l+1} \end{pmatrix}.$$

Then a fundamental solution matrix for the equation $\partial(\mathbf{y}) = A\mathbf{y}$ is a Wronskian matrix $W(y_1, \dots, y_{l+1}) =: Y \in \mathrm{GL}_{l+1}(E)$. Further, an inductive argument shows that the coefficient a_{l+1} of A satisfies

$$a_{l+1} = \frac{\partial(\det(Y))}{\det(Y)}.$$

Now, for $C \in \mathrm{SL}_{l+1}(C)$ we have that $\det(YC) = \det(Y)$, i.e., $\det(Y)$ is invariant under the action of the differential Galois group. It follows that $\det(Y) =: f$ is an element of F .

We show that the equation $\partial(\mathbf{y}) = A\mathbf{y}$ is differentially equivalent to an equation

$\partial(\mathbf{y}) = \bar{A}\mathbf{y}$ with defining matrix

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & \dots & & 0 & 1 \\ f_1 & f_2 & \dots & f_l & 0 \end{pmatrix}.$$

In a first step we show that \bar{A} is differentially equivalent to an element of the Lie algebra of $\mathrm{SL}_{l+1}(F)$. For this purpose, let B_1 be the diagonal matrix $B_1 := \mathrm{diag}(1, \dots, 1, \frac{1}{f})$. Simple matrix multiplications show

$$\mathrm{Ad}(B_1)(A) + l\delta B_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & & \\ \vdots & & & 1 & 0 \\ 0 & \dots & & 0 & f \\ \frac{a_1}{f} & \frac{a_2}{f} & \dots & \frac{a_l}{f} & 0 \end{pmatrix} := A_1 \in \mathrm{Lie}(\mathrm{SL}_{l+1})(F).$$

Note that until now we have no assumptions on F . Thus, for every differential field F and every Picard Vessiot extension E/F with differential Galois group $\mathrm{SL}_{l+1}(C)$ the corresponding torsor \mathcal{Z} has an F -rational point, i.e., there is an equivalent equation such that the corresponding torsor is the trivial torsor.

The next step is a differential conjugation of A_1 by the diagonal matrix

$$B_2 := \mathrm{diag}\left(\left(\frac{1}{f}\right)^{\frac{1}{l+1}}, \dots, \left(\frac{1}{f}\right)^{\frac{1}{l+1}}, \left(\frac{1}{f}\right)^{-\frac{l}{l+1}}\right).$$

Note that we assumed here that the field F contains the $(l+1)$ -th root of f . We obtain

$$\mathrm{Ad}(B_2)(A_1) + l\delta(B_2) = \begin{pmatrix} -\frac{\partial(f)}{(l+1)f} & 1 & 0 & \dots & 0 \\ 0 & -\frac{\partial(f)}{(l+1)f} & \ddots & & \\ \vdots & & & 1 & 0 \\ 0 & \dots & & -\frac{\partial(f)}{(l+1)f} & 1 \\ a_1 & a_2 & \dots & a_l & \frac{l\partial(f)}{(l+1)f} \end{pmatrix} =: A_2.$$

Since A_2 is an element of the subspace

$$A_\Delta + \mathbf{H}(F) \oplus \bigoplus_{\delta \in \Phi^-} \mathrm{Lie}(\mathrm{SL}_{l+1})_\delta(F),$$

we can apply Proposition 4.1. This yields a matrix $B_3 \in \mathrm{SL}_{l+1}(F)$ such that

$$\mathrm{Ad}(B_3)(A_2) + l\delta(B_3) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & \dots & & 0 & 1 \\ f_1 & f_2 & \dots & f_l & 0 \end{pmatrix} =: \bar{A}$$

holds. Hence, the matrix $B_3 B_2 B_1 \in \mathrm{GL}_{l+1}(F)$ defines a differential isomorphism from E/F to a Picard Vessiot Extension \bar{E}/F for $\partial(\mathbf{y}) = \bar{A}\mathbf{y}$ and the map $\sigma : (t_1, \dots, t_l) \mapsto (f_1, \dots, f_l)$ is the required specialization.

We summarize our results.

Proposition 5.1. *Let F be a differential field with field of constants C and suppose F satisfies the following property: For all $f \in F$ the $(l+1)$ -th root of f lies in F , i.e., $\sqrt[l+1]{f} \in F$.*

Let E/F be a Picard Vessiot extension with defining matrix $A \in M_{l+1}(F)$ and differential Galois group $\mathrm{SL}_{l+1}(C)$. Then there exists a specialization $\sigma : F_1 \rightarrow F$ such that

$$L(y, \sigma(\mathbf{t})) = y^{(l+1)} - \sum_{i=1}^l \sigma(t_i) y^{(i-1)} = 0$$

defines a Picard Vessiot extension which is differentially isomorphic to E/F .

Remark 5. Note that an algebraically closed differential field \bar{F} with field of constants C satisfies automatically the condition in Proposition 5.1. This proves the addendum from the introduction.

REFERENCES

- [1] N. Elkies, *Linearized algebra and finite groups of Lie type: I: Linear and symplectic groups*, pages 77-107 in Applications of curves over finite fields (Seattle, 1997) = Contemp. Math. 245, Providence: AMS, 1999.
- [2] L. Goldmann, *Specialization and Picard-Vessiot theory*, Trans. Am. Math. Soc., 85:327-356, 1957.
- [3] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972.
- [4] L. Juan and A. Ledet, *On generic differential SO_n -extensions*, Proc. Amer. Math. Soc., 136:1145–1153, 2008.
- [5] J. Kovacic, *The inverse problem in the Galois theory of differential fields*, Ann. Math., 89:583-608, 1969.
- [6] J. Kovacic, *Cyclic vectors and Picard-Vessiot extensions*, technical report, Polifics, Inc., 1996.
- [7] C. Mitschi and M.F. Singer, *Connected groups as differential galois groups*, J. Algebra 184 (1996), 333-361.
- [8] M. van der Put and M. F. Singer, *Galois theory of linear differential equations*, Springer-Verlag, Berlin-Heidelberg-New York, 2003.
- [9] M. Seiß, *Root Parametrized Differential Equations*, Dissertation, Universität Kassel, Kassel, 2011.
- [10] J.-P. Serre, *Galois cohomology*, Springer-Verlag, Berlin Heidelberg, 1997.

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